Ground state of the electron in the magnetic field of a straight current

2001 J. Phys. A: Math. Gen. 34 653

(http://iopscience.iop.org/0305-4470/34/3/324)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 653–662 www.iop.org/Journals/ja PII: S0305-4470(01)16618-7

Ground state of the electron in the magnetic field of a straight current

V M Tkachuk and S I Vakarchuk

Ivan Franko Lviv National University, Chair of Theoretical Physics, 12 Drahomanov Street, Lviv UA-79005, Ukraine

E-mail: tkachuk@ktf.franko.lviv.ua

Received 24 August 2000, in final form 8 November 2000

Abstract

The Pauli Hamiltonian for the electron moving in the magnetic field of a straight current with an axially symmetric distribution of current density possesses $N = 3$ broken supersymmetry. We study the ground state energy for the electron moving in this field. The asymptotic behaviour of ground state energy for small total angular momentum is obtained.

PACS numbers: 1130J, 0365

1. Introduction

The motion of the electron in a magnetic field has been the subject of interest for a long time. The eigenvalue problem for the charged spin-$\frac{1}{2}$ particle in a constant homogeneous magnetic field was investigated for the first time by Landau in the early days of quantum mechanics and the exact expressions for eigenstates and eigenenergies were obtained. Later a number of electromagnetic fields for which the eigenvalue problem can be solved exactly [1] were found. We would like to point out the interesting paper [2] where the quantum motion of the electron in a rotating magnetic field was solved exactly. Nevertheless for one of the simplest possible magnetic field configurations, namely, that of the current-carrying wire, it is impossible to obtain the exact solution of the corresponding eigenvalue problem. The energy spectrum and eigenstates of the charged particle with spin $\frac{1}{2}$ in the magnetic field of the current-carrying wire was determined numerically in [3]. The classical motion of the charged particle in this field was studied in [4]. In contrast to the case of the charged particle the eigenvalue problem for the neutral spin-$\frac{1}{2}$ particle moving in the magnetic field of the current-carrying wire can be solved exactly in different ways: by using supersymmetry in coordinate space [5], by differential equation techniques [6] and by using supersymmetry in the momentum space [7]. Surprisingly, the energy spectrum in this case obeys the hydrogenic Rydberg formula.

An important aspect of the motion of the electron in the magnetic field is the realization of the supersymmetry (SUSY) in this case (see reviews [8–10]). It was shown that $N = 2$ SUSY is realized in the case of an arbitrary two-dimensional magnetic field $B_x = B_y = 0$, $B_z = B_z(x, y)$ and the three-dimensional one which possesses the following symmetry with respect to the inversion of coordinates $\boldsymbol{B}(-\boldsymbol{r}) = \pm\boldsymbol{B}(\boldsymbol{r})$ [8–12]. The field of the magnetic monopole is one of the examples where SUSY is realized in the three-dimensional case [11]. It was also shown that the electron motion on the surface orthogonal to the magnetic field possesses $N = 2$ SUSY [13]. In our recent papers [14, 15] and the papers by Nikitin [16, 17] new three-dimensional magnetic fields in which the motion of the electron is supersymmetrical were found. Another novel aspect lies in the fact that in the magnetic fields considered SUSY with two, three and four supercharges is realized. In particular, in [15] we showed that the motion of the electron in the magnetic field of a straight current with an axially symmetric distribution of the current density possesses $N = 3$ SUSY. This case just includes the magnetic field of the current-carrying wire. In [18] we studied the equation for the zero-energy ground state of the electron moving in the magnetic field of a straight current. It was shown that this equation does not have the square integrable solution and SUSY is broken. Thus in the considered case the ground state has a non-zero energy level.

The aim of this paper is to study the ground state and the corresponding energy level for the electron moving in the magnetic field of a straight current.

2. SUSY of the electron in the magnetic field of a straight current with axial symmetry

The Pauli Hamiltonian for the electron moving in the magnetic field reads

$$H = \frac{1}{2m}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right)^2 - g\frac{e\hbar}{4mc}\boldsymbol{\sigma B} \tag{1}$$

where σ_α are the Pauli matrices, $\boldsymbol{A}$ is the vector potential and $\boldsymbol{B} = \mathrm{rot}\boldsymbol{A}$ is the magnetic field. The Hamiltonian (1) can be rewritten in the following form:

$$H = Q_0^2 - (g-2)\frac{e\hbar}{4mc}\boldsymbol{\sigma B} \tag{2}$$

where

$$Q_0 = \frac{1}{\sqrt{2m}}\boldsymbol{\sigma}\left(\boldsymbol{p} - \frac{e}{c}\boldsymbol{A}\right). \tag{3}$$

Note that for the electron the gyromagnetic ratio g only slightly differs from 2; namely, $g = 2.0023$. It is worth stressing that taking into account the anomalous magnetic moment of the electron ($g > 2$) leads to the so-called anomalous electron trapping by the magnetic fields [19–22]. These problems are outside the scope of our paper. We put $g = 2$. In this case the Pauli Hamiltonian possesses SUSY and Q_0 is called the supercharge.

In this paper we consider the motion of an electron in the magnetic field of a straight current. Let us assume the current to be parallel to the z axis with an axially symmetric distribution of the current density. The vector potential in this case reads

$$A_x = A_y = 0 \qquad A_z = A(\rho) \tag{4}$$

where $\rho = \sqrt{x^2 + y^2}$.

Recently we showed that in this case the Pauli Hamiltonian possesses $N = 3$ SUSY [15]; namely, in additional to Q_0 we have two more supercharges

$$Q_1 = \mathrm{i}\sigma_x I_x Q_0 \qquad Q_2 = \mathrm{i}\sigma_y I_y Q_0 \tag{5}$$

where I_x and I_y are the inversion operators of x and y axes respectively. We can easily check that supercharges fulfil the following SUSY algebra:

$$\begin{aligned} &\{Q_\alpha, Q_\beta\} = 2\delta_{\alpha,\beta}H \qquad \alpha, \beta = 0, 1, 2 \\ &[Q_\alpha, H] = 0. \end{aligned} \tag{6}$$

As a result of the axial symmetry the z component of the total angular momentum $J_z = L_z + S_z$ is the integral of motion and commutes with the Hamiltonian, i.e. $[J_z, H] = 0$; here L_z is the z component of angular momentum, $S_z = \hbar\sigma_z/2$ is the z component of the spin-$\frac{1}{2}$ operator. In addition, J_z satisfies the following permutation relations:

$$[J_z, Q_0] = 0 \tag{7}$$

$$\{J_z, Q_1\} = \{J_z, Q_2\} = 0. \tag{8}$$

In this paper we consider the case $g = 2$ when the Pauli Hamiltonian possesses SUSY and can be written as the squared supercharge Q_0. This fact is essentially used in the next sections for solving the eigenvalue problem (see equation (21)). It is necessary to stress that the motion of an electron in the magnetic field of a straight current possesses SUSY with three supercharges. However, in fact, to solve the eigenvalue problem we use only one of them, Q_0. Nevertheless, due to the fact of existence of N SUSY one can say that the degeneracy of the non-zero energy levels is equal to $2^{[N/2]}$, where square brackets mean the integer part of the number [12]. Thus, in our case ($N = 3$) the non-zero energy levels are twofold degenerate.

3. The eigenvalue problem

Due to the axial symmetry it is convenient to rewrite the Hamiltonian in the polar coordinates

$$H = -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{2m}\frac{1}{\rho^2}\left(-\mathrm{i}\hbar\frac{\partial}{\partial\phi}\right)^2 + \frac{1}{2m}\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - \frac{e}{c}A(\rho)\right)^2 - g\frac{e\hbar}{4mc}\boldsymbol{\sigma B}. \tag{9}$$

The coupling of the spin with magnetic field depends on ϕ

$$\boldsymbol{\sigma B} = A'(\rho)(\sigma_x \sin\phi - \sigma_y \cos\phi) \tag{10}$$

where $A'(\rho) = \partial A(\rho)/\partial\rho$. This dependence can be removed using the unitary transformation

$$\tilde{\psi} = \mathrm{e}^{\mathrm{i}\phi\sigma_z/2}\psi \tag{11}$$

where the new wavefunction satisfies the following condition:

$$\tilde{\psi}(\phi + 2\pi) = -\tilde{\psi}(\phi). \tag{12}$$

As a result of the unitary transformation equation (10) becomes

$$\mathrm{e}^{\mathrm{i}\phi\sigma_z/2}\boldsymbol{\sigma B}\mathrm{e}^{-\mathrm{i}\phi\sigma_z/2} = -A'(\rho)\sigma_y. \tag{13}$$

For the Pauli Hamiltonian after the unitary transformation we obtain

$$\tilde{H} = \mathrm{e}^{\mathrm{i}\phi\sigma_z/2}H\mathrm{e}^{-\mathrm{i}\phi\sigma_z/2} = -\frac{\hbar^2}{2m}\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{2m}\frac{1}{\rho^2}\left(-\mathrm{i}\hbar\frac{\partial}{\partial\phi} - \hbar\frac{\sigma_z}{2}\right)^2 + \frac{1}{2m}\left(-\mathrm{i}\hbar\frac{\partial}{\partial z} - \frac{e}{c}A(\rho)\right)^2 + g\frac{e\hbar}{4mc}A'(\rho)\sigma_y. \tag{14}$$

Note that after the unitary transformation the operator $-\mathrm{i}\hbar\frac{\partial}{\partial\phi} = \tilde{J}_z$ represents a new operator of the z component of the total angular momentum.

We can separate variables and represent the wavefunction as follows:

$$\tilde{\psi} = \mathrm{e}^{\mathrm{i}j_z\phi}\mathrm{e}^{\mathrm{i}kz}\frac{1}{\sqrt{\rho}}R(\rho) \tag{15}$$

where k is the wavevector of the electron motion along the z axis and $j_z = \pm\frac{1}{2}, \pm\frac{3}{2}, \ldots$ is the eigenvalue of the z component of the total angular momentum in the units of $\hbar$. The radial part $R(\rho)$ of the eigenfunction satisfies the conditions

$$R(0) = R(\infty) = 0. \tag{16}$$

Note that $R(\rho)$ must tend to zero at $\rho \to 0$ at least as $\sqrt{\rho}$ or more quickly. Then the wavefunction $\tilde{\psi}$ will be finite at $\rho = 0$.

The equation for $R(\rho)$ reads

$$H_\rho R(\rho) = E R(\rho) \tag{17}$$

where the radial part of the Pauli Hamiltonian (14) has the form

$$H_\rho = \frac{\hbar^2}{2m}\left(-\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho^2}\left[\left(j_z - \frac{\sigma_z}{2}\right)^2 - \frac{1}{4}\right] + \left(k - \frac{e}{\hbar c}A(\rho)\right)^2 + \frac{e}{\hbar c}A'(\rho)\sigma_y\right). \tag{18}$$

It is convenient to rewrite equation (17) and the radial part of the Hamiltonian (18) in dimensionless form. Let us put $A(\rho) = A_0 a(\rho/\rho_0)$, where the constant $A_0 > 0$ has the same dimension as the vector potential, $a(\rho/\rho_0)$ is the dimensionless function, ρ_0 is the unit of measurement of distance ρ. We choose $\rho_0 = \hbar c/|e|A_0$. Then in a dimensionless form equation (17) reads

$$H_\xi R(\xi) = \epsilon R(\xi) \tag{19}$$

where

$$H_\xi = -\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi^2}\left[\left(j_z - \frac{\sigma_z}{2}\right)^2 - \frac{1}{4}\right] + (\kappa + a(\xi))^2 - a'(\xi)\sigma_y. \tag{20}$$

We have introduced the notations $\epsilon = 2mE\rho_0^2/\hbar^2$, $\kappa = k\rho_0$ and $\xi = \rho/\rho_0$, which are dimensionless energy, wavevector and distance, respectively. Here $a'(\xi) = \partial a(\xi)/\partial\xi$.

The Hamiltonian H_ξ can be written in the form

$$H_\xi = Q^2 \tag{21}$$

where

$$\begin{aligned} Q &= -\mathrm{i}\sigma_x \frac{\partial}{\partial\xi} + f\sigma_z + \frac{j_z}{\xi}\sigma_y \\ f &= \kappa + a(\xi). \end{aligned} \tag{22}$$

Note that this is a result of equation (2) for $g = 2$ when the Pauli Hamiltonian possesses SUSY. The operator Q is the radial part of the supercharge Q_0 written in the polar coordinates.

It is important that the radial part of the Hamiltonian in the SUSY case ($g = 2$) is the squared operator Q. Then the eigenvalue problem reads

$$QR(\xi) = qR(\xi) \tag{23}$$

and the energy levels are squared eigenvalues of equation (23) $\epsilon = q^2$.

Consider the representation of the Pauli matrices for which σ_y is the diagonal matrix, namely

$$\sigma_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \sigma_z = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_x = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}. \tag{24}$$

Explicitly equation (23) is a set of two equations for the components R_1 and R_2 of the radial part of the wavefunction

$$\frac{j_z}{\xi}R_1 + \left(-\frac{\partial}{\partial\xi} + f\right)R_2 = qR_1 \tag{25}$$

$$\left(\frac{\partial}{\partial\xi} + f\right)R_1 - \frac{j_z}{\xi}R_2 = qR_2. \tag{26}$$

This set of the first-order differential equations can be transformed into the second-order differential equation for one of the components of the wavefunction. Just representation (24) is convenient for this purpose. From (26) we obtain

$$R_2 = \frac{1}{j_z/\xi + q}\left(\frac{\partial}{\partial\xi} + f\right) R_1. \tag{27}$$

In order to avoid a singularity we assume that j_z and q have the same signs. Then, because $\xi \geqslant 0$, the dominator in (27) cannot equal zero.

When j_z and q have the opposite signs it would be convenient to express R_1 over R_2, but it is not necessary to do this in an explicit form. For this, note that changing

$$q \to -q \qquad \kappa \to -\kappa \qquad a(\xi) \to -a(\xi) \tag{28}$$

leads to $R_1 \to R_2$ and $R_2 \to R_1$. Therefore in order to obtain the case for which j_z and q have the opposite signs it is necessary to apply (28) to the case for which j_z and q have the same signs.

Substituting (27) into (25) we obtain the following equation for R_1:

$$\left(-\frac{\partial}{\partial\xi} + f + u\right)\left(\frac{\partial}{\partial\xi} + f\right) R_1 + \frac{j_z^2}{\xi^2} R_1 = q^2 R_1 \tag{29}$$

where we have introduced the notation

$$u = -\frac{j_z/\xi^2}{j_z/\xi + q}. \tag{30}$$

Equation (29) is defined on the half-line $\xi \geqslant 0$ with the boundary conditions (16). As we see, the operator acting on R_1 in (29) is non-Hermitian and contains the first-order derivative. This operator can be transformed to the Hermitian form and the first-order derivative will be eliminated using the substitution

$$R_1 = \mathrm{e}^{\int \mathrm{d}\xi\, u/2} F_1(\xi) = \sqrt{j_z/\xi + q}\, F_1(\xi). \tag{31}$$

Then the equations for $F_1(\xi)$ can be written in the form

$$\left[a^+ a^- + \frac{j_z^2}{\xi^2}\right] F_1 = q^2 F_1 \tag{32}$$

where

$$a^\pm = \mp\frac{\partial}{\partial\xi} + f + u/2. \tag{33}$$

Note that F_1 satisfies the boundary conditions

$$F_1(0) = F_1(\infty) = 0 \tag{34}$$

and equation (32) similarly to (29) is defined on the half-line. In order to satisfy (16) $F_1(\xi)$ must tend to zero at $\xi \to 0$ at least as ξ or more quickly.

Thus, we reduced the set of two second-order differential equations (19) to the one second-order differential equation (32). This became possible due to equation (21), where the radial part of the Pauli Hamiltonian is written as a squared first-order differential operator.

We are interested in the square integrable solutions of equation (32). This equation has no square integrable solutions for $q = 0$. Indeed, the eigenvalues of the operator a^+a^-, which acts in the space of the square integrable functions, are positive, including also the zero eigenvalue. Thus, the eigenvalues of the operator $a^+a^- + j_z^2/\xi^2$ are positive without the zero eigenvalue. Therefore, equation (32) has no square integrable solutions for $q = 0$. Thus, we can conclude that for the electron moving in the magnetic field of a straight current the SUSY is broken and the energy of the ground state is non-zero.

4. Ground state: exact solution for the vector potential $1/\rho$

Let us first consider the case $a(\xi) = -\gamma/\xi$ for which the eigenvalue problem can be solved exactly [1]. The sign '−' is written for convenience. We consider this exactly solvable case to verify on its basis the result obtained in section 5.

The Hamiltonian in this case reads

$$H_\xi = -\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi^2}\left[\left(j_z - \frac{\sigma_z}{2}\right)^2 - \frac{1}{4} - \gamma\sigma_y\right] + \left(\kappa - \frac{\gamma}{\xi}\right)^2. \tag{35}$$

The spin and coordinate variables in the eigenvalue problem equation can be separated and the eigenstate can be written in the form

$$R(\xi) = \chi\varphi(\xi) \tag{36}$$

where $\varphi(\xi)$ satisfies the equation

$$\left[-\frac{\partial^2}{\partial\xi^2} + \frac{\lambda}{\xi^2} + \left(\kappa - \frac{\gamma}{\xi}\right)^2\right]\varphi(\xi) = \epsilon\varphi(\xi) \tag{37}$$

and χ satisfies the equation

$$\left[\left(j_z - \frac{\sigma_z}{2}\right)^2 - \frac{1}{4} - \gamma\sigma_y\right]\chi = \lambda\chi. \tag{38}$$

From the last equation we obtain

$$\lambda = \lambda_\pm = j_z^2 \pm \sqrt{\gamma^2 + j_z^2}. \tag{39}$$

Equation (37) is the textbook equation, which can be solved exactly using differential equation techniques or the factorization (supersymmetric) method. After factorization equation (37) reads

$$b^+b^-\varphi(\xi) + \epsilon_0\varphi(\xi) = \epsilon\varphi(\xi) \tag{40}$$

where

$$b^\pm = \left(\mp\frac{\partial}{\partial\xi} + \alpha - \frac{\beta}{\xi}\right). \tag{41}$$

α, β and the energy of factorization ϵ_0 satisfy the following equations:

$$\beta(\beta - 1) = \gamma^2 + \lambda \tag{42}$$

$$\alpha\beta = \kappa\gamma \tag{43}$$

$$\epsilon_0 = \kappa^2 - \alpha^2. \tag{44}$$

Equation (42) has two solutions, but only one of them

$$\beta = \tfrac{1}{2} + \sqrt{\tfrac{1}{4} + \gamma^2 + \lambda} \tag{45}$$

gives the necessary boundary condition (16) for the wavefunction at $\xi = 0$.

From equation (40) for the wavefunction of the ground state $\varphi_0(\xi)$ with the energy ϵ_0 we have the equation $b^-\varphi_0(\xi) = 0$, the solution of which is

$$\varphi_0(\xi) = C\xi^\beta e^{-\alpha\xi}. \tag{46}$$

Solution (45) gives $\varphi_0(0) = 0$, which satisfies (16). In order to satisfy the condition $\varphi_0(\infty) = 0$ we must choose $\alpha = k\gamma/\beta > 0$.

For the energy of the ground state with the fixed j_z and fixed spin state which is given by equation (38) we obtain

$$\epsilon_0 = \kappa^2 \left(1 - \frac{\gamma^2}{(1/2 + \sqrt{1/4 + \gamma^2 + \lambda})^2} \right). \tag{47}$$

Note that λ takes two values, given by (39), which correspond to two spin states. As we see from (47), ϵ_0 takes the lowest energy value for $\lambda = \lambda_-$. Thus, finally for the energy of the ground state with the fixed total angular momentum j_z we obtain

$$\epsilon_0 = \kappa^2 \left(1 - \frac{\gamma^2}{(1/2 + \sqrt{1/4 + \gamma^2 + j_z^2 - \sqrt{\gamma^2 + j_z^2}})^2} \right) \tag{48}$$

where $j_z = \pm 1/2, \pm 3/2, \ldots$.

Let us consider j_z as a parameter and put it equal to zero. It is interesting to note that in this case and for $|\gamma| \geqslant 1/2$ the ground state energy takes the smallest possible value $\epsilon_0 = 0$. The asymptotic behaviour of the ground state energy for small j_z and $|\gamma| \geqslant 1/2$ is the following:

$$\epsilon_0 = \frac{\kappa^2}{\gamma^2} j_z^2 \qquad j_z \to 0. \tag{49}$$

Note that really j_z cannot take zero value. Therefore, the energy of the ground state is non-zero. This coincides with the result obtained at the end of section 3.

5. Energy of the ground state for small j_z

In this section we derive the ground state energy in the limit of $j_z \to 0$ for the case of an arbitrary vector potential $a(\xi)$. It is possible to do this using equation (32) obtained in section 3. The second term j_z^2/ξ^2 for a small j_z in this equation can be treated as a perturbation.

In the zero approximation, i.e. when the term of perturbation is neglected, the energy of the ground state can be equal to zero. The function F_1 in this case satisfies the equation $aF_1 = 0$ and reads

$$F_1 = \frac{C}{\sqrt{j_z/\xi + q}} \mathrm{e}^{-\int f(\xi)\,\mathrm{d}\xi} = \frac{C}{\sqrt{j_z/\xi + q}} \mathrm{e}^{-\kappa\xi - \int a(\xi)\,\mathrm{d}\xi}. \tag{50}$$

Then

$$R_1 = C\mathrm{e}^{-\int f(\xi)\,\mathrm{d}\xi} = C\mathrm{e}^{-\kappa\xi - \int a(\xi)\,\mathrm{d}\xi}. \tag{51}$$

Using (27) we obtain that in the zero approximation $R_2 = 0$. We suppose that the vector potential is such that R_1 satisfies the imposed boundary conditions. Note that this result ($R_2 = 0$) is expected from equations (25) and (26), where in the zero approximation $j_z = 0$. Then the equations for R_1 and R_2 for zero-energy ground state ($q = 0$) read $a^- R_1 = 0$ and $a^+ R_2 = 0$, respectively. It is well known from SUSY quantum mechanics that only one of these equations has a square integrable solution. We consider such a vector potential $a(\xi)$ and wavevector κ that the square integrable solution exists for R_1. Then $R_2 = 0$.

In the first-order perturbation theory for the energy of the ground state ($\epsilon_0 = q^2$) we obtain

$$q^2 = \frac{\int_0^\infty \mathrm{d}\xi\, F_1^2 j_z^2/\xi^2}{\int_0^\infty \mathrm{d}\xi\, F_1^2} \tag{52}$$

or explicitly

$$q^2 = j_z^2 \int_0^\infty \mathrm{d}\xi\, \frac{1/\xi^2}{j_z/\xi + q} \mathrm{e}^{-2\kappa\xi - 2\int a(\xi)\,\mathrm{d}\xi} \Bigg/ \int_0^\infty \mathrm{d}\xi\, \frac{1}{j_z/\xi + q} \mathrm{e}^{-2\kappa\xi - 2\int a(\xi)\,\mathrm{d}\xi}. \tag{53}$$

This is the equation for q. As we see, q is proportional to j_z. Therefore we can write

$$q = \omega j_z \qquad j_z \to 0 \tag{54}$$

where ω satisfies the equation

$$\omega^2 = \int_0^\infty \mathrm{d}\xi\, \frac{1/\xi^2}{1/\xi + \omega} \mathrm{e}^{-2\kappa\xi - 2\int a(\xi)\,\mathrm{d}\xi} \Big/ \int_0^\infty \mathrm{d}\xi\, \frac{1}{1/\xi + \omega} \mathrm{e}^{-2\kappa\xi - 2\int a(\xi)\,\mathrm{d}\xi}. \tag{55}$$

In order to verify this result let us consider the vector potential $a(\xi) = -\gamma/\xi$ for which the exact solution for the ground state energy was obtained in the previous section. Substituting this vector potential in (50) for the function F_1 in the zero approximation of the perturbation theory we obtain

$$F_1 = C \frac{\xi^\gamma}{\sqrt{j_z/\xi + q}} \mathrm{e}^{-\kappa\xi} \tag{56}$$

and respectively

$$R_1 = C\xi^\gamma \mathrm{e}^{-\kappa\xi}. \tag{57}$$

This function satisfies the boundary conditions (16) when $\gamma > 1/2$ and $\kappa > 0$. Equation (55) in this case can be written in the form

$$1 = \int_0^\infty \mathrm{d}x\, \frac{x^{2\gamma-1}}{1+x} \mathrm{e}^{-px} \Big/ \int_0^\infty \mathrm{d}x\, \frac{x^{2\gamma+1}}{1+x} \mathrm{e}^{-px} \tag{58}$$

where we have introduced a new variable of integration $x = \omega\xi$ and the notation $p = 2\kappa/\omega$. We reduce this equation with respect to p to the following one:

$$1 = \frac{1}{2\gamma(2\gamma+1)} \frac{\Gamma(1-2\gamma, p)}{\Gamma(-1-2\gamma, p)} \tag{59}$$

where $\Gamma(a, x)$ is the incomplete Γ-function. We can verify that $p = 2\gamma$ is the solution of this equation and thus $\omega = \kappa/\gamma$. As a result for the ground state energy we obtain the same result (49) as in section 4. Thus, we can conclude that the general result (54), (55) for the ground state energy is correct. Note that using (28) we can obtain the result for the case $\gamma < -1/2$.

We conclude that the asymptotic behaviour of the ground state energy for small j_z for the electron moving in the magnetic field with the vector potential $a(\xi)$ is the following:

$$\epsilon_0 = q^2 = \omega^2 j_z^2 \qquad j_z \to 0 \tag{60}$$

where ω satisfies the equation (55).

Note that this result is applicable for such vector potentials $a(\xi)$ for which the functions F_1 or R_1 satisfy the necessary boundary conditions (34) or (16) and for which the integrals in (55) exist. This takes place, for instance, for

$$a(\xi) = -\gamma/\xi^{1+\delta} \tag{61}$$

where $\delta > 0$, $\gamma > 0$. Substituting this vector potential into (55) we obtain the equation for ω. In the case $\gamma < 0$ it is necessary to use (28).

For the vector potential of the current-carrying wire $a(\xi) = \gamma\ln(\xi)$ the obtained result is not applicable, because F_1 or R_1 in this case does not satisfy the necessary conditions at $\xi = 0$ and the integrals in (55) diverge in the vicinity of this point. This means that in this case the Hamiltonian (20) does not have the zero-energy ground state for $j_z = 0$.

6. Conclusions

In this paper we study the ground state for the electron moving in the magnetic field of a straight current with the axially symmetric distribution of the current density. The motion of the electron in this magnetic field possesses SUSY with three supercharges. We show that SUSY is broken, i.e. the energy of the ground state is non-zero.

An important point of this paper is that in the case ($g = 2$) when the Pauli Hamiltonian possesses SUSY we reduced the set of two second-order differential equations (19) for the eigenvalue problem to one second-order differential equation (32). This gives a possibility for an arbitrary vector potential $a(\xi)$ (with some restriction in order to satisfy the necessary boundary condition for the wavefunction) to derive an asymptotic behaviour of the ground state energy for a small total angular momentum $\epsilon_0 = \omega^2 j_z^2$, where ω satisfies equation (55). This result is applicable for the case of such a vector potential $a(\xi)$ for which Hamiltonian (20) (or equation (32)) at $j_z = 0$ has the zero-energy eigensolution. Note that here we treat j_z as a parameter which tends to zero although in fact it is the total angular momentum which takes the value $j_z = \pm 1/2, \pm 3/2, \ldots$. Therefore the ground state energy ϵ_0 does not reach the zero value.

The obtained asymptotic behaviour of the ground state energy takes place, for instance, in the case of $a(\xi) = -\gamma/\xi$, where $|\gamma| > 1/2$, and $a(\xi) = -\gamma/\xi^{1+\delta}$, where $\delta > 0$. Note, that in the case of the vector potential $a(\xi) = -\gamma/\xi$ the eigenvalue problem can be solved exactly. We present in this case the explicit expression for the energy and wavefunction of the ground state.

In conclusion, we would like to point out once more the motivation to consider the presented examples. The vector potential $a(\xi) = -\gamma/\xi$ is chosen from that point of view that it is an exactly solvable example and on its basis we can verify our method. The vector potential of the form (61) is the simplest example for which the proposed method can be applicable. Nevertheless, note that the obtained result is applicable for all vector potentials $a(\xi)$ for which the Hamiltonian (20) (or equation (32)) at $j_z = 0$ has the zero-energy eigenstate and for which the integrals in (55) converge.

For the case of the vector potential of the current-carrying wire $a(\xi) = \gamma \ln(\xi)$ the obtained result for the asymptotic behaviour of the ground state energy at small total angular momentum is not applicable, but we can definitely state that for arbitrary vector potentials produced by the straight current with the axially symmetric distribution of current density the energy of the ground state is positive and cannot take zero value.

References

[1] Bagrov V G, Gitman D M, Ternov I M, Halilov V P and Shapovalov V N 1982 *Exact Solutions of the Relativistic Wave Equation* (Novosibirsk: Nauka) p 144 (in Russian)
[2] Mielnik B and Fernandez D 1989 *J. Math. Phys.* **30** 537
[3] Müller M and Dietrich K 1996 *Phys. Rev.* A **54** 2577
[4] Müller M and Dietrich K 1995 *Z. Phys.* D **33** 101
[5] Hau L V, Golovchenko J A and Burns M M 1995 *Phys. Rev. Lett.* **75** 1426
[6] Blumel R and Dietrich K 1991 *Phys. Rev.* A **43** 22
[7] Voronin A I 1991 *Phys. Rev.* A **43** 29
[8] Gendenshteyn L E and Krive N V 1985 *Usp. Fiz. Nauk* **146** (4) 553
[9] Cooper F, Khare A and Sukhatme U 1995 *Phys. Rep.* **251** 267
[10] Junker G 1996 *Supersymmetric Methods in Quantum and Statistical Physics* (Berlin: Springer)
[11] D'Hoker E and Vinet L 1984 *Phys. Lett.* B **137** 72
[12] Gendenshteyn L E 1985 *Jadern. Fiz.* **41** (5) 261
[13] Sitenko Yu A 1990 *Jadern. Fiz.* **51** (5) 1416

[14] Tkachuk V M and Vakarchuk S I 1996 *J. Phys. Stud.* **1** 39
[15] Tkachuk V M and Vakarchuk S I 1997 *Phys. Lett.* A **228** 141
[16] Nikitin A 1996 *10th Int. Conf. on Problem of Quantum Field Theory* JINR E2-96-369
[17] Nikitin A 1997 *Group21, Physical Applications and Mathematical Aspects of Geometry, Groups and Algebras* vol 1, ed H-D Doebner, V Sherer and P Natterman (Singapore: World Scientific) p 509
[18] Tkachuk V M and Vakarchuk S I 1999 *J. Phys. Stud.* **3** 291
[19] Cavalkanti R M, Fraga E S and de Carvalho C A A 1997 *Phys. Rev.* B **56** 9243
[20] Bentosela F, Exner P and Zagrebnov V A 1998 *J. Phys. A: Math. Gen.* **31** L305
[21] Cavalcanti R M and de Carvalho C A A 1998 *J. Phys. A: Math. Gen.* **31** 7061
[22] Bentosela F, Cavalcanti R M, Exner P and Zagrebnov V A 1999 *J. Phys. A: Math. Gen.* **32** 3029 (Bentosela F, Cavalcanti R M, Exner P and Zagrebnov V A 1999 *Preprint* math-ph/9903030)